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A Pearson Random Walk with Steps of Uniform Orientation and Dirichlet Distributed Lengths

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Abstract A constrained diffusive random walk of n steps in $\mathbb{R}^d$ and a random flight in $\mathbb{R}^d$, which are equivalent, were investigated independently in recent papers (J. Stat. Phys. 127:813, 2007; J. Theor. Probab. 20:769, 2007, and J. Stat. Phys. 131:1039, 2008). The n steps of the walk are independent and identically distributed random vectors of exponential length and uniform orientation. Conditioned on the sum of their lengths being equal to a given value l, closed-form expressions for the distribution of the endpoint of the walk were obtained altogether for any n for $d = 1, 2, 4$. Uniform distributions of the endpoint inside a ball of radius l were evidenced for a walk of three steps in 2D and of two steps in 4D.

The previous walk is generalized by considering step lengths which have independent and identical gamma distributions with a shape parameter $q > 0$. Given the total walk length being equal to 1, the step lengths have a Dirichlet distribution whose parameters are all equal to q. The walk and the flight above correspond to $q = 1$. Simple analytical expressions are obtained for any $d \geq 2$ and $n \geq 2$ for the endpoint distributions of two families of walks whose q are integers or half-integers which depend solely on d. These endpoint distributions have a simple geometrical interpretation. Expressed for a two-step planar walk whose $q = 1$, it means that the distribution of the endpoint on a disc of radius 1 is identical to the distribution of the projection on the disc of a point M uniformly distributed over the surface of the 3D unit sphere. Five additional walks, with a uniform distribution of the endpoint in the inside of a ball, are found from known finite integrals of products of powers and Bessel functions of the first kind. They include four different walks in $\mathbb{R}^3$, two of two steps and two of three steps, and one walk of two steps in $\mathbb{R}^4$. Pearson–Liouville random walks, obtained by distributing the total lengths of the previous Pearson–Dirichlet walks according to some specified probability law are finally discussed. Examples of unconstrained random walks, whose step lengths are gamma distributed, are more particularly considered.

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1 Introduction

Pearson turned the problem of modelling the spatio-temporal evolution of the density of Anopheles mosquitoes in the jungle clearings into a simple planar "random walk", an expression coined for the first time in 1905 [1, 2]. The Pearson random walk is uncorrelated and unbiased: the direction of movement is completely independent of the previous directions moved and the direction moved at each step of constant length is completely random [1, 3–6]. The Pearson random walk and its variants find numerous applications in diverse fields such as physics, biology, ecology ([3–7] and references therein). Among the variants of the Pearson walk with unequal step sizes, a walk with shrinking step lengths was investigated very recently in 2D (the length of the ith step is $\lambda^{i-1}(\lambda < 1)$) [8]. Interestingly, the existence of a critical value, $\lambda_c = 0.5753882(3)$, is evidenced. The endpoint distribution changes from having a global maximum away from the origin of the walk for $\lambda < \lambda_c$ to being peaked at the origin for $\lambda > \lambda_c$ [8]. A large family of variants includes correlated random walks, which involve a correlation between successive step orientations. They are used to model the movement of animals, micro-organisms and cells, the dispersal of animals [4–7].

In this paper, by contrast, we focus on uncorrelated and unbiased variants of the Pearson random walk in Euclidean spaces $\mathbb{R}^d$ [9–11] and on random flights performed by a particle in $\mathbb{R}^d$ which are equivalent to the latter walks [12, 13]. We shall consider from now on only the cases where the lengths of the steps of the walks and the displacements between two changes of orientations of the flights are either gamma distributed or have Dirichlet distributions which are directly defined from gamma laws (Sect. 2).

1.1 Notations

A random variable X is recalled to be gamma distributed, with a shape parameter $q > 0$ and a scale parameter ρ, if its probability density function (pdf) $p(x)$ is equal to $x^{q-1}e^{-x/\rho}/(\rho^q\Gamma(q))$ $(x > 0)$, where $\Gamma(q)$ is the Euler gamma function. As the scale parameter is irrelevant in the present context, its value is fixed at 1. The considered walks and flights are thus specified by three parameters (d, n, q). The following notations are used throughout the rest of the text:

(1) d is the dimension of the Euclidean space in which the walk or the flight takes place.
(2) n is the number of steps and $m = n - 1$ is the number of reorientations.
(3) q is the shape parameter which characterizes the gamma distribution of the step length or its daughter distribution: the Dirichlet distribution (Sect. 2).
(4) lower cases are used to designate the probability density functions of the endpoint position while upper cases are used for the pdf's of the distance between the starting point and the endpoint. All pdf's are labelled by the ordered triplet (d, n, q). For gamma distributed step lengths (Sect. 8), the pdf's are denoted as:

   - $g_{d,n,q}(\boldsymbol{r})$, where $g_{d,n,q}(\boldsymbol{r})d\boldsymbol{r}$ represents the probability to find the endpoint of the n-step walk, or the position of the flying particle after m reorientations, within a small volume element $d\boldsymbol{r}$ at a point $\boldsymbol{r}$.

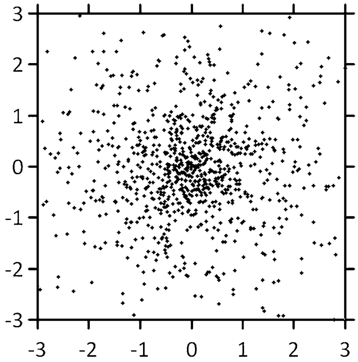

Fig. 1 Monte-Carlo simulation of the positions of the endpoints of 1000 independent planar random walks of two steps (the most distant walkers are not shown). The step orientations are uniform and independent, the step lengths s_k $(k = 1, 2)$ are exponentially distributed, $p(s_k) = \exp(-s_k)$, $(q = 1)$, and independent

– $G_{d,n,q}(r)$, where $G_{d,n,q}(r)dr$ represents the probability to find the latter endpoint or the latter position at a distance from the starting point ranging between r and $r + dr$.
  Similarly, $p_{d,n,q}(\boldsymbol{r})$ and $P_{d,n,q}(r)$ denote the corresponding pdf's in the case of Dirichlet distributed step lengths.

(5) unconstrained step lengths are denoted as s_k $(k = 1, .., n)$ while step lengths whose sum is constrained to have a given value l are denoted as l_k $(k = 1, .., n)$, $\sum_{k=1}^{n} l_k = l$.

1.2 Random Walks and Random Flights with $q = 1$

To model the motion of microorganisms on planar surfaces, Stadje [9] investigated a 2D random flight. A microorganism starts at the origin, moves in straight-line paths at constant speed, and changes its direction after exponentially distributed time intervals. Paths have independent uniform orientations and independent and identically distributed (i.i.d.) lengths. Stadje derived the exact pdf of the position $\boldsymbol{r}$ of the microorganism at time t and the conditional pdf of its position at the time of the nth turn (see further Sect. 8.2). Figure 1 shows the positions of the endpoints of walkers which perform an isotropic planar random walk of two steps with exponentially distributed lengths ($d = 2$, $n = 2$, $q = 1$).

Considering a particle which moves in a random environment and undergoes elastic collisions at uniformly distributed point obstacles, Franceschetti [10] defined a variant of the previous walk in $\mathbb{R}^d$: as above, the n steps of the walk are i.i.d. d-dimensional random vectors whose lengths have an exponential distribution ($q = 1$) and whose orientations are uniform. However, the total travelled length is constrained to be equal to a given l (> 0), whose value is fixed here at 1 without loss of generality. All walks of more than one step end thus on a disc of radius 1 centered at the starting point (Figs. 2 and 3). Figure 2 ($n = 2$) illustrates the result of fixing to 1 the total length of the two-step planar walk of Fig. 1.

Franceschetti derived the conditional pdf of the position of the endpoint $\boldsymbol{r}$ for a walk in $\mathbb{R}^d$ $(d = 1, 2)$ of n steps whose total length is fixed. The latter pdf is denoted consistently $p_{d,n,1}(\boldsymbol{r})$ as we will show in Sect. 2 that the step length distribution is actually a Dirichlet distribution with $q = 1$. The pdf $p_{d,n,1}(\boldsymbol{r})$ depends only on the distance $r = \|\boldsymbol{r}\|$ between the starting point and the endpoint as the walk is statistically invariant by any orthogonal transformation. In 2D, the pdf $p_{2,n,1}(\boldsymbol{r})$ reads [10]:

$$p_{2,n,1}(\boldsymbol{r}) = \frac{(n-1)}{2\pi}(1 - r^2)^{\frac{n-3}{2}} \quad (r < 1)\ n = 2, 3, \ldots \tag{1}$$

Franceschetti concluded that a walker is more likely to end its walk near the boundary of the disc of radius 1 when making less than three steps in 2D (Fig. 2 for $n = 2$) and near the

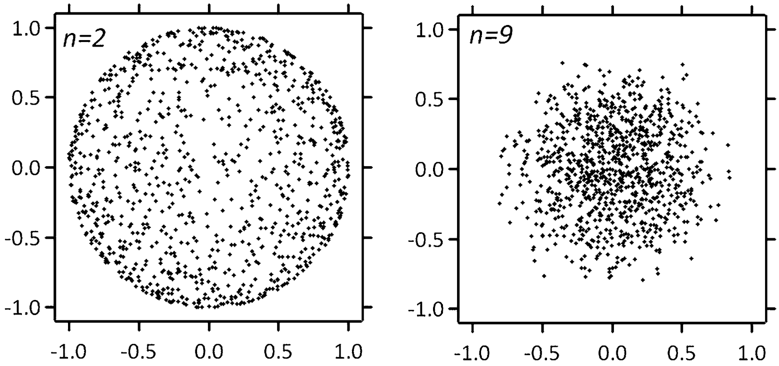

Fig. 2 Monte-Carlo simulation of the positions of the endpoints of 1000 independent Pearson–Dirichlet planar random walks of n steps ($n = 2, 9$), $PD(d = 2, n, q = 1)$

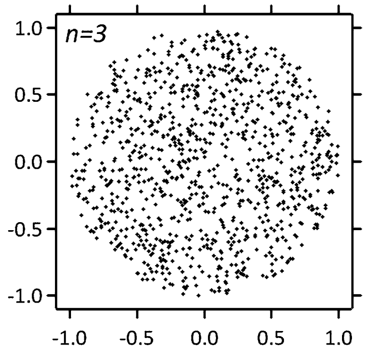

Fig. 3 Monte-Carlo simulation of the positions of the endpoints of 1000 independent Pearson–Dirichlet planar random walks of three steps, $PD(d = 2, n = 3, q = 1)$. The walkers are uniformly distributed on a disc of radius 1

origin when making more than three steps (Fig. 2 for $n = 9$). By making exactly three steps (Fig. 3), the endpoint is uniformly distributed inside the disc of radius 1 (1). A geometrical interpretation of (1) is given in Fig. 4 for $n = 2$. The value $n = 3$ in 2D was considered as a 'critical transition point' in the behaviour of the random walk. The question naturally arose as to whether it is possible to find a uniform distribution of endpoints for another couple (d, n). As calculations were thought to become intractable in dimensions higher than $d = 2$, Franceschetti [10] derived a necessary condition for endpoints to be uniformly distributed inside a ball of radius l $(= 1)$. From the second moment of the distance of the endpoint to the origin, he obtained a relation between n and d, $d(n - 1) = 4$, which is only satisfied by the two couples ($d = 2$, $n = 3$) and ($d = 4$, $n = 2$). The Franceschetti walk was later formulated in terms of scattering of particles by García-Pelayo [11]. He concluded that the ($d = 2$, $n = 3$) walk is the sole walk of the whole family whose endpoint distribution is uniform.

As shown in Sect. 2, the aforementioned walk is equivalent to a random flight performed by a particle in $\mathbb{R}^d$ which starts from the origin at time $t = 0$, moves with a constant and finite velocity c in an initial random direction. It flies at a constant velocity c until it chooses instantaneously a new direction, at a random time determined by a homogeneous Poisson process, independently of the previous direction [12, 13]. For a given time interval t, the total length of the flight is then fixed, $l = ct$. The value $q = 1$ results here from the exponential distribution of the time interval between two successive Poisson events (Sect. 2). The conditional pdf of the position of the particle at time t, given the number m of Poisson events

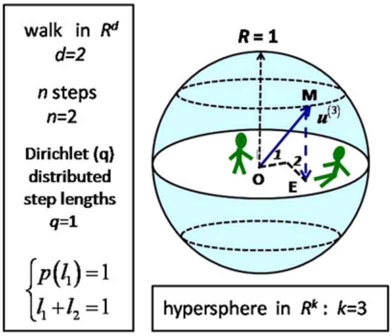

Fig. 4 (Color online) The "hyperspherical uniform" property: a planar two-step walk, whose total length is 1, starts at O in a random direction and ends at E. The distribution of E on a disc of unit radius centered at O ((1) with $n = 2$) is identical to the distribution of the projection on the disc of a point M uniformly distributed over the surface of the 3D unit sphere. (inspired by a representation of the celestial sphere: Observatoire de Paris http://media4.obspm.fr/)

that occurred up to t, was obtained for any m for $d = 2$ and for $d = 4$ by Orsingher and De Gregorio [12] and by Kolesnik [13]. The pdf $p_{4,n,1}(\boldsymbol{r})$ follows from their results:

$$p_{4,n,1}(\boldsymbol{r}) = \frac{n(n-1)}{\pi^2}(1-r^2)^{n-2} \quad (r < 1)\ n = 2, 3, \ldots \tag{2}$$

A uniform distribution exists then for $n = 2$ in 4D [12, 13] in contradiction with the conclusion of García-Pelayo [11]. The latter is explained by an error in equation (7) of [11] in which $4!/6!$ must be replaced by $(2+s)!/(4+s)!$.

1.3 The "Hyperspherical Uniform" Property for $q = 1$

A simple geometrical interpretation is hidden behind the pdf's given by (1) and (2). It is illustrated in Fig. 4 for the case of the previous two-step planar walk (1). The distribution of the endpoint on a disc of unit radius, centered at the starting point, is identical to the distribution of the projection on the disc of a point M uniformly distributed over the surface of the 3D unit sphere. From now on, we will name for brevity "hyperspherical uniform" (HU) a walk which possesses the latter property. Its endpoint distribution is then identical to the distribution of the projection in the walk space $\mathbb{R}^d$ of a point, with a position vector $\boldsymbol{u}^{(k)}$, randomly chosen on the surface of the unit hypersphere of some hyperspace $\mathbb{R}^k$ ($k = 3$ in Fig. 4). The pdf's $p_{2,n,1}(\boldsymbol{r})$ (1) and $p_{4,n,1}(\boldsymbol{r})$ (2) are easily obtained in that way from (59) of the Appendix with $k = n + 1$ when $j = d = 2$ and with $k = 2n + 2$ when $j = d = 4$ respectively ([13] and Sect. 3). The existence of uniform distributions for $d = 2, 4$ stems actually from the HU property (end of Appendix). We realized independently of [13] that the constrained planar walk of Sect. 1.2 is hyperspherical uniform (emails were exchanged on that topic with M. Franceschetti (6–7 June 2007)). This observation motivated the present work.

1.4 The Step Length Distribution

The step lengths of the random walks and random flights of Sect. 1.2 are uniformly distributed over the unit $(n-1)$ simplex (Sect. 2.1). This distribution is also a Dirichlet distribution whose parameters are all equal to 1. More generally, we will consider "Pearson–Dirichlet" random walks of n steps in $\mathbb{R}^d$ denoted by $PD(d,n,q)$ whose parameters are chosen to be all equal to a given positive value q (Sect. 2). A simple way to describe the latter walks is to consider step lengths which have a gamma distribution with a shape parameter q. Given a total walk length equal to 1, the step lengths have then a Dirichlet distribution whose parameters are all equal to q. Monte-Carlo simulations of $PD(2,n,1)$ walks are shown in Figs. 2 and 3 for $n = 2, 3, 9$.

1.5 Aims and Sketch of the Method

The aims of the present work are to find all hyperspherical uniform walks among the Pearson–Dirichlet walks $PD(d,n,q)$ and to determine the associated hyperspace dimensions k. The pdf's of the endpoints of such HU walks, $p_{d,n,q}(\boldsymbol{r})$, are then readily obtained from the dimensions k by replacing simply j by d in (59). The derivation of their analytical expressions, $p_{d,n,q}(\boldsymbol{r}) \propto (1-r^2)^{\delta}$, reduces then to that of their exponents $\delta(d,n)$. Such walks are notable because very simple closed-form expressions do exist for the pdf's of the final locations of the walkers. The possible occurrence of other uniform distributions, $p_{d,n,q}(\boldsymbol{r}) = \text{constant} \Leftrightarrow \delta(d,n) = 0$, can then be examined. The method may be summarized as follows:

(1) restrict the search for HU walks to Pearson–Dirichlet walks $PD(d,n,q)$ only
(2) find a necessary condition for a walk of this family to be HU and deduce the parameters (d,n,q) and the hyperspace dimension k of all the walks which might be HU
(3) from a recurrence relation obeyed by the characteristic functions of the distributions of the endpoint positions, prove that the selected HU walks are all HU
(4) from the pdf's of the HU walks, infer cases when the endpoint is uniformly distributed in the inside of the unit hypersphere of $\mathbb{R}^d$

Besides the two uniform walks in 2D and in 4D reported for $q = 1$ [10, 12, 13], five additional uniform walks, four in 3D and one in 4D, will be shown to be associated with known finite integrals of products of powers and Bessel functions of the first kind.

A Pearson–Liouville random walk will be finally defined from the Pearson–Dirichlet walks by letting the total walk length l vary according to some probability law. Relations between the pdf's of the endpoint of the unconstrained walk and the pdf's of the constrained one will be established. An illustrative example of random walks of n i.i.d. steps in $\mathbb{R}^d$, whose lengths are gamma distributed, will be considered.

2 Dirichlet Distribution and Pearson–Dirichlet Walks

2.1 The Uniform Distribution on the Unit $(n-1)$-simplex

The joint pdf $p_e(s_1, s_2, .., s_n)$ of i.i.d. exponential step lengths $s_i > 0$ $(i = 1, .., n)$, with a scale parameter of 1, is simply given by:

$$p_e(s_1, s_2, .., s_n) = \exp\left(-\sum_{i=1}^{n} s_i\right) \tag{3}$$

Defining first their sum as $s = \sum_{i=1}^{n} s_i$ and $l_i = s_i/s$ $(i = 1, .., n)$, then changing the set of variables from $(s_1, s_2, .., s_n)$ to $(l_1, l_2, .. l_m, s)$, $(m = n-1)$, the following joint pdf is obtained from the Jacobian of the transformation which is equal to s^m:

$$p_u(l_1, l_2, ..l_m, s) = m!\left(\frac{1}{m!}s^m \exp(-s)\right) = p(l_1, l_2, ..l_m) \times p_S(s) \tag{4}$$

Then $\boldsymbol{l}^{(m)} = (l_1, l_2, ..l_m)$ and s are independent. As expected, the distribution of s, which is the sum of $n = m + 1$ i.i.d. exponential random variables, is a gamma distribution with a shape parameter $q = n$. Further, the pdf, $p(l_1, l_2, .., l_m) = m!$, is constant. Rescaling, if necessary, the total length to make $l = \sum_{k=1}^{n} l_k = 1$, the distribution of the step length of the constrained walk described in Sect. 1.2 [10] is, by construction, uniform over the unit $(n-1)$ simplex $\boldsymbol{S}_{n-1}$ defined by:

$$\boldsymbol{S}_{n-1} = \left\{(l_1, l_2, .., l_n) \in \mathbb{R}^n : \sum_{i=1}^{n} l_i = 1 \text{ and } l_i \geq 0 \text{ for any } i\right\} \tag{5}$$

The same conclusion holds for the random flight considered in [12, 13]. Indeed, the joint pdf of the times of occurrence of events from an homogeneous Poisson process in the time interval $(0, t]$, given that the number of events is $N(t) = m$, is $p(t_1, t_2, .., t_m | N(t) = m) = \frac{m!}{t^m}(0 < t_1 < t_2 < \cdots < t_m \leq t)$ (see for instance [14] p. 277). The distribution of the inter-arrival times, in a time interval scaled down to unit length, is thus identical to the previous distribution $p(l_1, l_2, .., l_m)$ as the Jacobian of the transformation is 1. The inter-arrival time distribution $(t = 1)$ is also that of m i.i.d. random variables U_i $(i = 1, .., m)$ which are uniformly distributed on the unit interval (0–1) [14, 15]. When their values are sorted in increasing order, the ordered arrival "times" are denoted as $U_{(i)}$ $(i = 1, .., m)$. Then, the multivariate pdf of the inter-arrival times, $V_{(i)} = U_{(i)} - U_{(i-1)}$ $(i = 1, .., m;\ U_{(0)} = 0)$, is uniform over the unit $(n-1)$ simplex [15].

The distribution of the endpoint of the walk studied by Franceschetti [10] and the conditional distribution of the particle after a random flight of duration t investigated by Orsingher and De Gregorio [12] and by Kolesnik [13] are concluded to be identical after a replacement of ct by l ((1) and (2) with $l = 1$). For a total length of 1, the step lengths are uniformly distributed over the unit $(n-1)$ simplex, a distribution which is a particular case of a Dirichlet distribution whose parameters are all equal to $q = 1$ as shown by (6) below.

2.2 The Dirichlet Distribution

The Dirichlet distribution is of common use in simplices. It is applied for instance in ecology, to model fragmentation or compositional data [16, 17]. The walk performed by a donkey inside a tetrahedron, as constructed by Letac [18], leads to a stationary distribution which is Dirichlet. The Dirichlet distribution can be simply defined as follows [19]: consider a set of $n = m + 1$ independent gamma distributed random variables, s_i $(> 0,\ i = 1, .., n)$ with shape parameters $\alpha_i > 0$ $(i = 1, .., n)$ and scale parameters of 1 (for simplicity) and define $l_j = s_j / \sum_{i=1}^{n} s_i$ $(j = 1, .., n)$. The distribution of $\boldsymbol{l}^{(m)}$ is a Dirichlet distribution with a vector of parameters $\boldsymbol{\alpha}^{(n)} = (\alpha_1, \ldots, \alpha_m, \alpha_n)$, denoted as $D_m(\boldsymbol{\alpha}^{(n)})$. Using a method identical to the previous one, its pdf is established to be ([19], p. 17):

$$\begin{cases} p_m(l_1, .., l_m) = K(\boldsymbol{\alpha}^{(n)}) \prod_{i=1}^{n} l_i^{\alpha_i - 1} \\ l_n = 1 - \sum_{i=1}^{m} l_i, \quad l_i > 0,\ i = 1, .., n \end{cases} \tag{6}$$

where $K(\boldsymbol{\alpha}^{(n)}) = \Gamma(\alpha)/(\prod_{i=1}^{n}\Gamma(\alpha_i))$, $\alpha = \sum_{i=1}^{n}\alpha_i$. Defining a vector $\boldsymbol{\beta}^{(n)} = (\beta_1, \ldots, \beta_m, 0)$ and $\beta = \sum_{i=1}^{n}\beta_i$, the moment $M_{\boldsymbol{\beta}} = \langle\prod_{i=1}^{m} l_i^{\beta_i}\rangle$ is simply obtained by noticing that it is related to the normalization constant of the Dirichlet distribution $D_m(\boldsymbol{\alpha}^{(n)} + \boldsymbol{\beta}^{(n)})$, namely:

$$M_{\beta} = \left\langle \prod_{i=1}^{m} l_i^{\beta_i} \right\rangle = K(\boldsymbol{\alpha}^{(n)})/K(\boldsymbol{\alpha}^{(n)} + \boldsymbol{\beta}^{(n)}) = \left(\prod_{i=1}^{m}(\alpha_i)_{\beta_i}\right)/(\alpha)_{\beta} \tag{7}$$

where $(a)_r = \Gamma(a+r)/\Gamma(a)$ reduces to an ascending factorial, $(a)_r = a(a+1)..(a+r-1)$, when r is an integer. The Dirichlet distribution has a notable amalgamation property [19]. If the n components $l_1, .., l_n$ ($\sum_{i=1}^{n} l_i = 1$) of a vector, whose distribution is $D_m(\boldsymbol{\alpha}^{(n)})$, are grouped into k components $v_1, .., v_k$ ($\sum_{i=1}^{k} v_i = 1$), then the distribution of $(v_1, .., v_{k-1})$ is $D_{k-1}(\boldsymbol{\alpha}^{*(k)})$ where each α_i^* ($i = 1, .., k$) is the sum of the α_j's corresponding to the components of the initial vector which add up to v_i. The marginal distribution of any component l_i is then obtained by grouping the remaining components into a single one. Any component l_i has thus a beta distribution [20] with a pdf:

$$p_i(l_i) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_i)\Gamma(\alpha - \alpha_i)} \times l_i^{\alpha_i - 1}(1 - l_i)^{\alpha - \alpha_i - 1} \quad (0 < l_i < 1).$$

The amalgamation property results directly from the well-known fact that a sum of independent gamma random variables, with identical scale parameters and a priori different shape parameters, is still a gamma random variable with the same scale parameter and a shape parameter which is the sum of all shape parameters [19, 20].

The method based on the generation of i.i.d. gamma random variables is the simplest method for simulating Dirichlet distributions on simplices ([21] for the case $\boldsymbol{\alpha}^{(n)} = (1, .., 1)$) that was used in the present work for Monte-Carlo simulations of Pearson–Dirichlet random walks. Finally, it is readily seen from the definition of the Dirichlet distribution that the conditional distribution of $l_j^* = \frac{l_{j+1}}{1-l_1}$, ($j = 1, .., m-1$), given l_1, is still a Dirichlet distribution (Theorem 1.6 of Fang et al. [19]), $D_{m-1}(\boldsymbol{\alpha}^{(n-1)} = (\alpha_2, .., \alpha_{m,}\alpha_n))$, with a pdf:

$$p_{m-1}(l_1^*, .., l_{m-1}^*|l_1) = p_{m-1}(l_1^*, .., l_{m-1}^*) = K(\boldsymbol{\alpha}^{(n-1)})\left[\prod_{i=1}^{m} l_i^{*\alpha_{i+1}-1}\right] \tag{8}$$

As the distribution of $\boldsymbol{l}^{*(m-1)}$ is independent of the distribution of l_1, then l_1 is said to be neutral [22]. This property serves as a basis for a generalization of Dirichlet distributions ([23] and references therein). The step lengths, $(l_1, l_2, ..l_m)$, of the Pearson–Dirichlet walks $PD(d, n, q)$ considered in what follows have a Dirichlet distribution $D_m(\boldsymbol{q}^{(n)})$, where $\boldsymbol{q}^{(n)}$ is the n-dimensional vector whose components are all equal to q ($q > 0$). In this case, the distribution $p_{m-1}(l_1^*, .., l_{m-1}^*)$ is simply $D_{m-1}(\boldsymbol{q}^{(n-1)})$.

2.3 Pearson–Dirichlet Walks and Flights

When q is an integer, a Pearson–Dirichlet walk $PD(d, n, q)$ ($d \geq 2$) can be interpreted equally in terms of random walks similar to the walks described in Sect. 1.2 [10]: instead of changing its direction after every step with an exponentially distributed length, the particle changes it at every q steps, the intermediate steps being ineffective (Fig. 5). The distribution of the step length between two changes of direction is then the sum of q i.i.d. exponential random contributions, that is a gamma distribution with a shape parameter q.

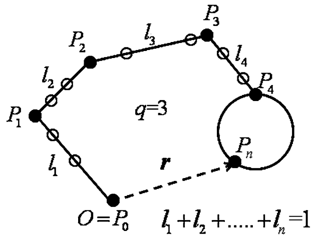

Fig. 5 A Pearson–Dirichlet random walk $PD\ (2, n, q)$ in $\mathbb{R}^2$ for $q = 3$: the walk starts at O in a random direction; the length of every step is the sum of q (as indicated by $q - 1$ empty circles) i.i.d. exponential random variables; at P_k $(k = 1, .., n - 1)$, a new random direction is taken independently of the previous ones; every step is rescaled so as to make the total travelled length equal to 1. The walk ends at P_n. Equivalently, the n step lengths are i.i.d. random variables with a gamma distribution whose shape parameter is q

Equivalently, we may consider that the n steps of the walk are i.i.d. d-dimensional random vectors, whose orientations are uniform, whose lengths have a gamma distribution with a pdf $p(s_k) = s_k^{q-1}\mathrm{e}^{-s}/\,\Gamma(q)$ $(k = 1, .., n)$, where q may have any positive value. The latter walk, given its total length being equal to 1, is a Pearson–Dirichlet walk whose step lengths have by definition a Dirichlet distribution $D_m(\boldsymbol{q}^{(n)})$ (Sect. 2.2).

Similarly, the conditional pdf of the times of occurrence of Poisson events in $(0, 1]$, given their number $N(1) = nq - 1$, is $p(t_1, t_2, .., t_{nq-1}|N(1) = nq - 1) = (nq - 1)!$ $(0 < t_1 < t_2 < \cdots < t_{nq-1} \leq 1)$ (Sect. 2.1, [14, 15]) and the inter-arrival times distribution is thus a Dirichlet distribution $D_{nq-1}(\boldsymbol{\alpha}^{(nq)} = (1, 1, \ldots, 1))$. Amalgamating the nq variables q by q gives the sought-after Dirichlet distribution $D_m(\boldsymbol{q}^{(n)})$. During its flight, a particle changes then its direction at every q Poisson events, $q - 1$ intermediate events being ineffective (Fig. 5). Beghin and Orsingher [24] studied recently a planar random flight of this type, $PD(d = 2, n, q = 2)$ (see also Sect. 8.2).The flight described above might be looked at as a persistent random flight with $q_e = 1$ and fully correlated orientations of q successive steps.

3 A Necessary Condition for a Walk $PD(d, n, q)$ to Be "Hyperspherical Uniform"

The general problem of obtaining closed-form expressions of the probability density function of the endpoint of a walk of n steps $PD(d, n, q)$ in $\mathbb{R}^d$ is intractable for any n and d. Explicit expressions of the pdf $p_{d,2,1}(\boldsymbol{r})$ were however obtained by Orsingher and De Gregorio (equation (2.25) of [12]) for the endpoints of walks of two steps $PD(d, 2, 1)$ in any dimension $d \geq 2$:

$$p_{d,2,1}(\boldsymbol{r}) = \frac{2^{d-3}\Gamma(d/2)}{\pi^{d/2}}(1 - r^2)^{(d-3)/2}F\left(d - 2, \frac{1}{2}; \frac{d}{2}; r^2\right) \quad (r < 1) \tag{9}$$

where $F(, ; ;)$ is a Gaussian hypergeometric function. The latter density reduces to an even polynomial in r of degree $d - 4$ for even $d \geq 4$. Kolesnik [25] derived explicit expressions of the pdf's $p_{6,n,1}(\boldsymbol{r})$ of walks of n steps in 6D which reduce to even polynomials of finite orders. It is for instance, an even polynomial of degree 6 in r for $n = 3$ (equation (15) of [25]).

Rather than coping with an essentially insoluble problem, we chose to search under the lamppost to find all Pearson–Dirichlet walks $PD(d, n, q)$ whose distributions of the endpoint $p_{d,n,q}(\boldsymbol{r})$ are simple to calculate. The conjunction of known finite integrals of products of two Bessel functions of the first kind with powers and the possible existence of recurrence relations for values of $q > 0$ other than 1 led us to select the hyperspherical uniform property as our lamppost.

The Dirichlet distribution of the step-length vector $\boldsymbol{l}^{(m)}$ has a multivariate pdf given by:

$$p_m(l_1, .., l_m) = \frac{\Gamma(nq)}{\Gamma(q)^n} \times \left[\prod_{i=1}^{n} l_i^{q-1}\right] \tag{10}$$

with $l_n = 1 - \sum_{i=1}^{m} l_i$, $l_i > 0$, $(i = 1, .., n)$. The endpoint of a Pearson–Dirichlet walk $PD(d, n, q)$ is a vector of $\mathbb{R}^d$ which reads $(n \geq 2)$:

$$\boldsymbol{r}_n^{(d)} = \sum_{i=1}^{n} l_i \boldsymbol{u}_i^{(d)} \tag{11}$$

where the $\boldsymbol{u}_i^{(d)} = (u_i(1), .., u_i(d))$, $(i = 1, .., n)$, are n independent unit vectors uniformly distributed over the surface of a hypersphere in $\mathbb{R}^d$. A simple necessary condition for a walk to be HU is that the even moments of a single component $r_1 = r_n(1)$ of $\boldsymbol{r}_n^{(d)}$ are equal to the even moments of any component of a unit vector uniformly distributed over the surface of a hypersphere in some space $\mathbb{R}^k$, where k has to be determined. This necessary condition will thus provide all possible sets (d, n, q, k) for which the sought-after property might hold. Actually, the moments $\langle r_1^2 \rangle$ and $\langle r_1^4 \rangle$ happen to suffice. The moment $\langle r_1^2 \rangle$ is just n times the product of $\langle l_i^2 \rangle = \frac{q(q+1)}{(nq)(nq+1)}$ by $\langle u_i(j)^2 \rangle = \frac{1}{d}$, the cross-products being zero because the $\boldsymbol{u}_i^{(d)}$'s are independent and have zero means. Similarly, the moment $\langle r_1^4 \rangle$ is given by the sum: $n\langle l_1^4 \rangle \langle u_1^4(1) \rangle + 3n(n-1)\langle l_1^2 l_2^2 \rangle \langle u_1^2(1) \rangle^2$. These moments are (7) and (60):

$$\begin{cases} \langle r_1^2 \rangle = \dfrac{1}{k} = \dfrac{q+1}{d(nq+1)} \\ \langle r_1^4 \rangle = \dfrac{3}{k(k+2)} = \dfrac{3(q+1)(q+2)(q+3)}{d(d+2)(nq+1)(nq+2)(nq+3)} + \dfrac{3(n-1)q(q+1)^2}{d^2(nq+1)(nq+2)(nq+3)} \end{cases} \tag{12}$$

We notice first that (12) yields the expected result for $n = 1$, namely $k = d$ as the endpoint of a walk of one step is, by definition, uniformly distributed over the surface of the hypersphere in $\mathbb{R}^d$. In the following, we take $n > 1$. The moment $\langle r_1^2 \rangle$ gives:

$$k = \frac{d(nq+1)}{q+1} \tag{13}$$

which, when plugged into $\langle r_1^4 \rangle$ (12), gives a relation between d, n and q that simplifies to a quadratic equation:

$$d^2 - 3(q+1)d + 2(q+1)^2 = 0 \tag{14}$$

which does not depend on n. The two solutions, whose correctness is readily verified from (12), and the corresponding "hyperspace" dimensions are:

$$\begin{cases} d' = q+1 & k' = nq+1 \\ d'' = 2(q+1) & k'' = 2(nq+1) \end{cases} \tag{15}$$

Table 1 Parameters needed to obtain the pdf of the endpoint $p_{d,n,q}(\boldsymbol{r})$, and that of the distance of the endpoint to the origin $P_{d,n,q}(r)$, for the two families of Pearson–Dirichlet walks $PD(d,n,q)$ which are HU

	Parameter of the Dirichlet distribution	Hypersphere in $\mathbb{R}^k$:	$p_{d,n,q}(\boldsymbol{r}) \propto (1-r^2)^\delta$ (59)
Family number $i=1,2$	$q=\frac{d}{i}-1$ $(d \geq i+1)$	$k=n(d-i)+i$	$\delta=\frac{(n-1)(d-i)-2}{2}$
$p_{d,n,q}(\boldsymbol{r})=\frac{\Gamma(k/2)}{\Gamma(\delta+1)\pi^{d/2}}(1-r^2)^\delta$			$P_{d,n,q}(r)=\frac{2\Gamma(\delta+d/2+1)}{\Gamma(\delta+1)\Gamma(d/2)}r^{d-1}(1-r^2)^\delta$

The previous necessary condition yields two possibilities: either $q(\geq 1/2)$ is an integer or it is a half-integer. Two families of HU walks, numbered one and two, $PD(d,n,q)$, are then found. For family i $(i=1,2)$, the parameter q and the hyperspace dimension k are given by Table 1. As required, k is equal to d for $n=1$. The HU walks, $PD(2,n,1)$ $(i=1, q=1)$ in 2D and $PD(4,n,1)$ $(i=2, q=1)$ in 4D, described in Sect. 1.2 (1) [12, 13], are then seen to belong to different HU families.

Using a recurrence relation between the characteristic functions of the probability distributions of the endpoints of walks of $n-1$ and of n steps, quite similar to that derived by Kolesnik [13] for $q=1$, we prove in the next section that the previous walks are indeed hyperspherical uniform walks for any n. The necessary conditions of the present section will thus be found to be sufficient. Table 1 gathers the parameters needed to obtain the endpoint pdf $p_{d,n,q}(\boldsymbol{r})$ from (59) and consequently that of the distance of the endpoint to the origin $P_{d,n,q}(r)$ for the two HU families of the considered Pearson–Dirichlet walks. The former pdf, given by $p_{d,n,q}(\boldsymbol{r}) \propto (1-r^2)^\delta$ (Table 1) reduces then to a constant when $d-i=2/(n-1)$ is an integer, that is for $d=i+1 (n=3)$ and $d=i+2 (n=2)$. Four walks whose endpoints are uniformly distributed in the inside of the unit hypersphere of $\mathbb{R}^d$ are therefore found in that way for two steps and for three steps. Two of them were previously described: $d=2$, $n=3$ and $d=4$, $n=2$ [10, 12, 13] (Sect. 1). Two are new: $d=3$, $n=2$ and $d=3$, $n=3$. Three additional uniform walks are obtained in Sect. 5 for random walks with Dirichlet distributions of step lengths whose parameters are not all the same. The distribution of the square of the distance, $s=r^2$, is a beta distribution with parameters $d/2$ and $\delta+1$. Once the latter pdf's are known for $l=1$, one gets immediately that:

$$\begin{cases} p^{(l)}_{d,n,q}(\boldsymbol{r}) = \dfrac{1}{l^d} p_{d,n,q}\left(\dfrac{\boldsymbol{r}}{l}\right) \\ P^{(l)}_{d,n,q}(r) = \dfrac{1}{l^d} P_{d,n,q}\left(\dfrac{r}{l}\right) \end{cases} \tag{16}$$

for an arbitrary total walk length l. Table 2 collects all the characteristics of the four previous uniform walks and those of the three additional uniform walks mentioned above.

4 A Recurrence Relation

To establish that a walk $PD(d,n,q)$ is hyperspherical uniform, it suffices to prove that the characteristic function (c.f.) of the probability distribution of the endpoint $\boldsymbol{r}=\boldsymbol{r}^{(d)}_n$ is $\Omega_k(\rho)$ (58). The latter c.f. is that of a unit vector whose tip is uniformly distributed over

Table 2 The seven Pearson–Dirichlet HU walks, with step length distributions $D_m(\boldsymbol{\alpha}^{(n)} = (\alpha_1, \alpha_2, .., \alpha_n))$, whose endpoints are uniformly distributed in the inside of a hypersphere in $\mathbb{R}^d$ (the first and the sixth walks are described in [10, 12, 13])

Walk in $\mathbb{R}^d$	Number of steps n	Dirichlet parameters $\boldsymbol{\alpha}^{(n)} = (\alpha_1, \alpha_2, .., \alpha_n)$	Step length pdf's $D_{n-1}(\boldsymbol{\alpha}^{(n)})$
$\mathbb{R}^2$	3	$(1, 1, 1)$	$p_2(l_1, l_2) = 2$ $(l_3 = 1 - l_1 - l_2)$ $p_1(l_i) = 2(1 - l_i)$ $(i = 1, ., 3)$
$\mathbb{R}^3$	2	$(2, 2)$	$p_1(l_1) = 6l_1(1 - l_1)$ $(l_2 = 1 - l_1)$
$\mathbb{R}^3$	2	$(2, 3)$	$p_1(l_1) = 12l_1(1 - l_1)^2$ $(l_2 = 1 - l_1)$
$\mathbb{R}^3$	3	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$p_2(l_1, l_2) = 1/(2\pi\sqrt{l_1 l_2 l_3})$ $(l_3 = 1 - l_1 - l_2)$ $p_1(l_i) = 1/(2\sqrt{l_i})$ $(i = 1, ., 3)$
$\mathbb{R}^3$	3	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$p_2(l_1, l_2) = (3/(2\pi)) \times \sqrt{l_1/(l_2 l_3)}$ $(l_3 = 1 - l_1 - l_2)$ $p_1(l_1) = 3\sqrt{l_1}/2$ $p_1(l_i) = 3(1 - l_i)/(4\sqrt{l_i})$ $(i = 2, 3)$
$\mathbb{R}^4$	2	$(1, 1)$	$p_1(l_1) = 1$ $(l_2 = 1 - l_1)$
$\mathbb{R}^4$	2	$(2, 1)$	$p_1(l_1) = 2l_1$ $(l_2 = 1 - l_1)$

the surface of the hypersphere in $\mathbb{R}^k$ with $k = n(d - i) + i$, $d \geq i + 1$, $i = 1, 2$. The conditional pdf of the rescaled step lengths, $p_{m-1}(l_1^*, l_2^*, ..l_{m-1}^* | l_1)$, which is a Dirichlet distribution whose parameters are all equal to q (8), allows us to express the endpoint of the walk of $n \geq 2$ steps in $\mathbb{R}^d$ as follows:

$$\boldsymbol{r}_n^{(d)} = l_1 \boldsymbol{u}_1^{(d)} + \left[\sum_{i=2}^{n} l_i \boldsymbol{u}_i^{(d)}\right] = l_1 \boldsymbol{u}_1^{(d)} + (1 - l_1)\left[\sum_{i=1}^{n-1} l_i^* \boldsymbol{u}_i^{(d)}\right] = l_1 \boldsymbol{u}_1^{(d)} + (1 - l_1)\boldsymbol{r}_{n-1}^{(d)} \quad (17)$$

From (17) and the marginal pdf, $p_1(l_1) = \frac{\Gamma(nq)}{\Gamma(q)\Gamma((n-1)q)} l_1^{q-1}(1 - l_1)^{(n-1)q-1}$, we obtain the c.f. of the probability distribution of $\boldsymbol{r}_n^{(d)}$, $\Phi_{d,n,q}(\rho) = \langle \exp(i\boldsymbol{\rho} \cdot \boldsymbol{r}_n^{(d)}) \rangle$ $(\rho = \|\boldsymbol{\rho}\|)$:

$$\begin{cases} \Phi_{d,1,q}(\rho) = \Omega_d(\rho) \\ \Phi_{d,2,q}(\rho) \propto \displaystyle\int_0^1 l_1^{q-1}(1 - l_1)^{q-1} \Omega_d(\rho l_1) \Omega_d(\rho(1 - l_1)) dl_1 \\ \Phi_{d,n,q}(\rho) \propto \displaystyle\int_0^1 l_1^{q-1}(1 - l_1)^{(n-1)q-1} \Omega_d(\rho l_1) \Phi_{d,n-1,q}(\rho(1 - l_1)) dl_1 \quad (n \geq 3) \end{cases} \quad (18)$$

We don't have to worry about the proportionality constants in (18), as their final values are simply obtained from the condition that $\Phi_{d,n,q}(0) = 1$ for any n. A walk of one step is HU, by definition, and its c.f. $\Phi_{d,1,q}(\rho)$ is therefore identical to $\Omega_d(\rho)$ for any $q > 0$ but Pearson–Dirichlet walks $PD(d, n, q)$ are not all hyperspherical uniform. We determine next the conditions for the HU property to hold for a walk of two steps. Using (58) and (18), it

Table 3 The parameters of the two families of HU Pearson–Dirichlet walks $PD(d,n,q)$, which are needed to solve the recurrence relations (22, 23) from integrals (20) and (21) (hypersphere in $\mathbb{R}^k$ with $k=a(n-1)+b$ for $n-1$ steps)

Family i $(=1,2)$ q	a	b	Integral equation no	μ	v ($n-1$ steps)
$\frac{d}{i}-1$ $(d \geq i+1)$	$d-i$	i	$19+i$	$\frac{d}{2}-1$	$\frac{(d-i)(n-1)+i-2}{2}$

comes:

$$\Phi_{d,2,q}(\rho) \propto \frac{1}{\rho^{2q-1}} \int_0^\rho x^{q-d/2}(\rho-x)^{q-d/2} J_{(d-2)/2}(x) J_{(d-2)/2}(\rho-x)dx \tag{19}$$

where $J_u(x)$ is a Bessel function of the first kind. The following finite integrals:

$$\int_0^\rho x^\mu(\rho-x)^v J_\mu(x) J_v(\rho-x)dx = \frac{\Gamma(\mu+1/2)\Gamma(v+1/2)}{\sqrt{2\pi}\,\Gamma(v+\mu+1)} \rho^{\mu+v+1/2} J_{\mu+v+1/2}(\rho) \tag{20}$$

$(\mu, v > -1/2)$ (integral 6.581.3 of [28]) and:

$$\int_0^\rho \frac{J_\mu(x) J_v(\rho-x)}{x(\rho-x)} dx = \frac{(\mu+v)}{\mu v \rho} J_{\mu+v}(\rho) \tag{21}$$

$(\mu, v > 0)$ ([29], p. 380) yield explicit expressions of integral (19) either when:

- $q = d-1$ ((20), $\mu+v+1/2 = (2d-3)/2$)
  or when:
- $q = (d-2)/2$ ((21), $\mu+v = d-2$)

The parameters $q = \frac{d}{i}-1$ $(i=1,2)$ derived in Sect. 3 are precisely those which obey the latter conditions. The walks of two steps, whose parameters are obtained from Table 1 for $n=2$, are then concluded from (19, 20, 21) and (58) to be hyperspherical uniform with $k=2d-i$, $(d \geq i+1, i=1,2)$. Let us assume now that the walks with parameters d, q, k given in Table 1, are HU for $(n-1) \geq 3$ steps, that is $\Phi_{d,n-1,q}(\rho) = \Omega_{a(n-1)+b}(\rho)$ where $a(=a(d))$ and b are reported separately in Table 3. Then, (18) writes $(n \geq 2)$:

$$\Phi_{d,n,q}(\rho) \propto \int_0^1 l_1^{q-1}(1-l_1)^{(n-1)q-1} \Omega_d(\rho l_1) \Omega_{a(n-1)+b}(\rho(1-l_1))dl_1 \tag{22}$$

which reduces to:

$$\Phi_{d,n,q}(\rho) \propto \frac{1}{\rho^{nq-1}} \int_0^\rho x^{q-d/2}(\rho-x)^{(n-1)(q-a/2)-b/2} J_{(d-2)/2}(x) J_{(a(n-1)+b-2)/2}(\rho-x)dx \tag{23}$$

From (23) and integrals (20) and (21), we deduce that $\Phi_{d,n,q}(\rho) = \Omega_{an+b}(\rho)$ for the walks whose parameters are given in Tables 1 and 3. As the explicit calculations are all performed in the same way, we will just present one of them and derive the c.f., $\Phi_{2j+1,n,q}(\rho) = \Omega_{(2j-1)n+2}(\rho)$, with $q=(2j-1)/2$ for a walk of n steps in $\mathbb{R}^{2j+1}$ which belongs to the second family. To obtain $\Phi_{2j+1,n,q}(\rho)$, we assume that $\Phi_{2j+1,n-1,q}(\rho) = \Omega_{(2j-1)(n-1)+2}(\rho)$ for a walk of $n-1$ steps $(n \geq 2)$ using the parameters deduced from Table 3 for $i=2$, $a=2j-1$, $b=2$. From the recurrence relation (23), it comes:

$$\Phi_{2j+1,n,q}(\rho) \propto \frac{1}{\rho^{(2j-1)n/2-1}} \int_0^\rho x^{-1}(\rho-x)^{-1} J_{(2j-1)/2}(x) J_{(2j-1)(n-1)/2}(\rho-x)dx \tag{24}$$

From (21) with $\mu = (2j-1)/2$ and $v = (2j-1)(n-1)/2$, we conclude that $\Phi_{2j+1,n,q}(\rho) \propto J_{(2j-1)n/2}(\rho)/\rho^{(2j-1)n/2}$ and from (58) and the condition, $\Phi_{2j+1,n,q}(0) = 1$, we obtain finally $\Phi_{2j+1,n,q}(\rho) = \Omega_{(2j-1)n+2}(\rho)$ which proves that the HU property holds for n when it holds for $n-1 (n \geq 2)$.

In sum, we have shown that any walk of n steps defined in Table 1 is HU given that it is HU for $(n-1)$ steps $(n \geq 2)$. As the property holds for $n = 1$ (and for $n = 2$) it holds for any n. We conclude that the walks evidenced by the necessary condition of Sect. 3 are all HU for any n. The corresponding parameters and distributions of the endpoint are given in Table 1. Two families of HU Dirichlet walks exist in any space $\mathbb{R}^d$ with $d \geq 3$ and only one family for $d = 2$.

5 Additional Hyperspherical Uniform Walks Among Pearson–Dirichlet Random Walks

Two other finite integrals of products of powers and Bessel functions of the first kind yield additional HU walks.

A HU walk of two steps in any space of dimension d greater than 1 is indeed obtained for the following Dirichlet distribution $D_1(\boldsymbol{\alpha}^{(2)} = (d-1, d))$ (and by symmetry $D_1(\boldsymbol{\alpha}^{(2)} = (d, d-1))$):

$$p_1(l_1) = \frac{(2d-2)!}{(d-1)!(d-2)!} l_1^{d-2}(1-l_1)^{d-1} \tag{25}$$

Then, from $\boldsymbol{r}_2^{(d)} = l_1 \boldsymbol{u}_1^{(d)} + (1-l_1)\boldsymbol{u}_2^{(d)}$, we write the characteristic function (the different parameters of the Dirichlet distribution are now specified in the notation of the c.f.):

$$\Phi_{d,2,(d-1,d)}(\rho) = \langle \exp(i\boldsymbol{\rho} \cdot \boldsymbol{r}_2^{(d)}) \rangle \propto \int_0^1 l_1^{d-2}(1-l_1)^{d-1}\Omega_d(\rho l_1)\Omega_d(\rho(1-l_1))dl_1 \tag{26}$$

that is:

$$\Phi_{d,2,(d-1,d)}(\rho) \propto \frac{1}{\rho^{2d-2}} \int_0^\rho x^{d/2-1}(\rho-x)^{d/2} J_{d/2-1}(x) J_{d/2-1}(\rho-x)dx \tag{27}$$

From $\Phi_{d,2,(d-1,d)}(0) = 1$ and the following integral:

$$\int_0^\rho x^\mu (\rho-x)^{v+1} J_\mu(x) J_v(\rho-x)dx = \frac{\Gamma(\mu+1/2)\Gamma(v+3/2)}{\sqrt{2\pi}\Gamma(v+\mu+2)} \rho^{\mu+v+3/2} J_{\mu+v+1/2}(\rho) \tag{28}$$

$(\mu > -1/2, v > -1)$ (integral 6.581.4 of [28]), and from $\mu + v + 1/2 = (2d-3)/2$, we get (58):

$$\Phi_{d,2,(d-1,d)}(\rho) = \Omega_{2d-1}(\rho) \tag{29}$$

The latter walk is then concluded to be HU with $k = 2d-1$. The distribution of the endpoint is finally obtained from (59) with $j = d$:

$$p_{d,2,\{d-1,d\}}(\boldsymbol{r}) = \frac{\Gamma((2d-1)/2)}{\Gamma((d-1)/2)\pi^{d/2}} (1-r^2)^{(d-3)/2} \tag{30}$$

and consequently:

$$P_{d,2,\{d-1,d\}}(r) = \frac{2^{d-1}\Gamma((2d-1)/2)}{(d-2)!\sqrt{\pi}} r^{d-1}(1-r^2)^{(d-3)/2} \tag{31}$$

A fifth HU walk, whose endpoint is uniformly distributed in the inside of a sphere in $\mathbb{R}^3$, is then found for a walk of two steps.

The results of the previous paragraph extend to a walk in $\mathbb{R}^d$ whose step lengths have a Dirichlet distribution $D_m(\boldsymbol{\alpha}^{(n)} = (d, d-1, d-1, .., d-1))$. Except for the first, the Dirichlet parameters of this HU walk coincide with those of the first HU family as does the hyperspace dimension, $k = n(d-1)+1$.

A last family of HU walks is found in any space of dimension d greater than 2 from the following integral ($\mu > 0,\ v > -1$) ([29] p. 380):

$$\int_0^\rho \frac{J_\mu(x) J_v(\rho - x)}{x} dx = \frac{1}{\mu} J_{\mu+v}(\rho) \tag{32}$$

The associated Dirichlet distribution is, $D_1(\boldsymbol{\alpha}^{(2)} = (\frac{d}{2}-1, \frac{d}{2}))$, for a two-step walk with a distribution of l_1 given by:

$$p_1(l_1) = \frac{(d-2)!}{\Gamma((d-2)/2)\Gamma(d/2)} l_1^{(d-4)/2} (1-l_1)^{(d-2)/2} \tag{33}$$

As above, we write the characteristic function from $\boldsymbol{r}_2^{(d)} = l_1 \boldsymbol{u}_1^{(d)} + (1-l_1)\boldsymbol{u}_2^{(d)}$:

$$\Phi_{d,2,(d/2-1,d/2)}(\rho) \propto \frac{1}{\rho^{d-2}} \int_0^\rho \frac{J_{(d-2)/2}(x) J_{(d-2)/2}(\rho - x)}{x} dx \tag{34}$$

and we get from (32) and (58):

$$\Phi_{d,2,(d/2-1,d/2)}(\rho) = \Omega_{2d-2}(\rho) \tag{35}$$

from which the latter walk is concluded to be HU with $k = 2d-2$. The c.f. of a walk of three steps, with a step length distribution given by $D_2(\boldsymbol{\alpha}^{(3)} = (\frac{d}{2}, \frac{d}{2}-1, \frac{d}{2}-1))$, writes similarly:

$$\Phi_{d,3,\{d/2,d/2-1,d/2-1\}}(\rho) \propto \frac{1}{\rho^{3(d-2)/2}} \int_0^\rho \frac{J_{(d-2)/2}(x) J_{d-2}(\rho - x)}{x} dx \propto \frac{J_{3(d-2)/2}(\rho)}{\rho^{3(d-2)/2}} \tag{36}$$

That is, $\Phi_{d,3,\{d/2,d/2-1,d/2-1\}}(\rho) = \Omega_{3d-4}(\rho)$. As before, these results hold for any walk in $\mathbb{R}^d$ ($d > 2$) whose step lengths have a Dirichlet distribution $D_m(\boldsymbol{\alpha}^{(n)} = (d/2, d/2-1, d/2-1, .., d/2-1))$. The latter walk is then HU with a hyperspace dimension $k = n(d-2)+2$ which coincides with that of the second HU family. The distribution of the endpoint and that of the distance from the endpoint to the origin are therefore obtained from Table 1 with $\delta = (n(d-2)-d)/2$. Thus, two uniform walks, which satisfy the condition $n = d/(d-2)$, are obtained for the couples ($d = 3,\ n = 3$) with $\boldsymbol{\alpha}^{(3)} = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ and ($d = 4,\ n = 2$) with $\boldsymbol{\alpha}^{(2)} = (2, 1)$. As seen in Table 2, these walks occur in the same spaces and with the same number of steps than two previous walks but their step length distributions are different.

In sum, five additional 'uniform' walks were found in the present work: two walks of two steps and two of three steps in $\mathbb{R}^3$ and one of two steps in $\mathbb{R}^4$. Table 2 collects the characteristics of the seven uniform Pearson–Dirichlet walks. The uniform walks found in $\mathbb{R}^3$ and in $\mathbb{R}^4$ are seen to be degenerate as they are obtained for more than one set of Dirichlet parameters. Monte-Carlo simulations of the pdf's of these five walks were found to be in perfect agreement with the calculated ones (see Fig. 6 for $d = 3$, $n = 3$, $q = 1/2$).

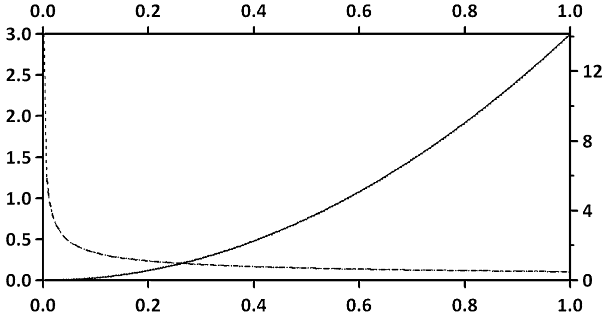

Fig. 6 Monte-Carlo simulations (10^8 walks) of a Pearson–Dirichlet random walk of three steps in $\mathbb{R}^3$ with $q = 1/2$ (the step lengths are obtained from the squares of three independent standard Gaussians whose sum is normalized to 1). The endpoints are uniformly distributed in the inside of a sphere of radius 1. The differences between the simulated pdf's and the calculated ones are typically less than the thicknesses of the lines. The pdf of the distance from the endpoint to the origin (*left* axis) is $P_{3,3,1/2}(r)$ (Table 1, $d = 3$, $n = 3$, $q = 1/2$) and the marginal pdf of a step length (*right* axis) is $p_1(l_i)$ (Table 2)

6 Stochastic Representation of Hyperspherical Uniform Walks

A simple stochastic representation of the endpoint $\boldsymbol{r}$ exists for a HU walk. We define a k-dimensional Gaussian vector $\boldsymbol{G}^{(k)}$, $N(\boldsymbol{0}, \boldsymbol{I}_k)$ where $\boldsymbol{I}_k$ is the unit $k \times k$ matrix, whose components are independent random variables with zero means and variances of 1. When normalized, it yields a unit vector $\boldsymbol{u}^{(k)} = \boldsymbol{G}^{(k)}/\|\boldsymbol{G}^{(k)}\|$ whose tip is uniformly distributed over the surface of the unit hypersphere in $\mathbb{R}^k$ (Appendix). The square of the modulus of $\boldsymbol{G}^{(k)}$, $\chi_k^2 = \|\boldsymbol{G}^{(k)}\|^2$, follows, by definition, a chi-square distribution with k degrees of freedom [20]. It is too a gamma distribution with a shape parameter of $k/2$ and a scale parameter of $1/2$. It can be split into two independent chi-square random variables: $\chi_k^2 = \chi_d^2 + \chi_{k-d}^2$ with respective degrees of freedom d and $k - d$. To obtain the endpoint of the walk $\boldsymbol{r}_n^{(d)}$, it suffices to retain the first d components of $\boldsymbol{u}^{(k)}$. Therefore:

$$\boldsymbol{r}_n^{(d)} \triangleq \frac{\boldsymbol{G}^{(d)}}{\sqrt{\|\boldsymbol{G}^{(d)}\|^2 + \chi_{k-d}^2}} \tag{37}$$

where $\boldsymbol{G}^{(d)}$ is now a d-dimensional Gaussian vector, $N(\boldsymbol{0}, \boldsymbol{I}_d)$, and where, by convention, $\chi_0^2 = 0$. In (37), $a \triangleq b$ means that a and b are identically distributed. The vector $\boldsymbol{G}^{(d)}$ and the random variable χ_{k-d}^2 are independent. Similarly, the distance of the endpoint to the origin is represented by:

$$r_n^{(d)} \triangleq \frac{\chi_d}{\sqrt{\chi_d^2 + \chi_{k-d}^2}} \tag{38}$$

(χ_d has a chi distribution [20] with d degrees of freedom). Equations (37) and (38) were used to perform fast Monte-Carlo simulations of any HU walk. It suffices indeed to generate $d + 1$ independent random variables to simulate the endpoint positions $\boldsymbol{r}_n^{(d)}$ and only two to simulate $r_n^{(d)}$ for any n and d.

7 Asymptotic Behavior

For $n = 1$, the endpoints of any Pearson–Dirichlet walk $PD(d,n,q)$ are uniformly distributed over the surface of a unit d-dimensional hypersphere. When n increases for a fixed d, the endpoints invade progressively the inner part of the hypersphere forming a spherically symmetric cloud for any n (Figs. 2 and 3). When $n \to \infty$, the latter cloud shrinks gradually into a Gaussian spherical cloud which is more and more concentrated around the origin. In all cases, $\langle \|\boldsymbol{r}_n^{(d)}\|^2 \rangle = d\langle r_1^2 \rangle = \frac{q+1}{(nq+1)}$ (12), decreases regularly with n independently of d. For the HU walks, the latter scenario is a direct consequence of a theorem of Diaconis and Freedman [30] which proves that the first d coordinates of a point uniformly distributed over the surface of a $k = an + b$ sphere are independent standard normal variables, in the limit as $n \to \infty$ with d fixed.

For any dimension d, the main term which contributes to the moment $\langle r_1^{2p} \rangle$ of a single component of $\boldsymbol{r}_n^{(d)}$ in the limit as $n \to \infty$, is $\frac{(2p)!}{2^p} \times \frac{n(n-1)..(n-p+1)}{p!} \times \langle l_1^2 l_2^2 .. l_p^2 \rangle \times \langle u_1^2(1) \rangle^p$. From (7) and (10), it comes:

$$\langle r_1^{2p} \rangle_\infty = \lim_{n\to\infty} \frac{(2p)!}{2^p d^p} \times \frac{n(n-1)..(n-p+1)}{p!} \times \frac{(q(q+1))^p}{\prod_{j=1}^{2p}(nq+j-1)} = (2p-1)!! \left(\frac{q+1}{nqd} \right)^p \tag{39}$$

which are the moments of a Gaussian distribution with a zero mean and a variance equal to $\frac{q+1}{nqd}$. As the distribution of $\boldsymbol{r}_n^{(d)}$ is spherically symmetric, the latter argument indicates that the Gaussian behavior holds in the asymptotic limit for any walk $PD(d, n \to \infty, q)$ with d fixed.

8 Pearson–Liouville Random Walks

8.1 Definition and Generalities

The Pearson–Dirichlet walk $PD(d,n,q)$, and specifically the HU walks whose parameters are given in Table 1, can serve as "unit" walks to mix walks with different total lengths l. We assume then that the total length l is distributed according to some continuous probability density function $f(l)$ and we denote the new step lengths as $\boldsymbol{s}^{(n)} = (s_1, .., s_n)$ ($\sum_{k=1}^n s_k = l$). The renormalized step lengths, $\boldsymbol{v}^{(n)} = (l_1 = s_1/l, .., l_n = s_n/l)$ ($\sum_{k=1}^n l_k = 1$) have a Dirichlet distribution whose parameters are all equal to q for any value of l (10). Then the joint pdf of $\boldsymbol{s}^{(n)}$ is:

$$p_L(s_1, .., s_n) = \frac{\Gamma(nq)}{\Gamma(q)^n} \times \prod_{k=1}^n s_k^{q-1} \times \frac{f\left(\sum_{k=1}^n s_k\right)}{\left(\sum_{k=1}^n s_k\right)^{nq-1}} \tag{40}$$

The step length distribution, given by (40), is a Liouville distribution with a generating density $f(\cdot)$ (Chap. 6 of [19, 31]). The associated random walk will be named consistently a Pearson–Liouville random walk. The stochastic representation of the step length vector is $\boldsymbol{s}^{(n)} \triangleq l\boldsymbol{v}^{(n)}$, where l and $\boldsymbol{v}^{(n)}$ are independent. If the generating density is defined in a finite interval $(0, L)$, then $p_L(s_1, .., s_n)$ is defined in the simplex $((s_1, .., s_n) : \sum_{k=1}^n s_k \le L)$. If $f(\cdot)$ is defined on $\mathbb{R}^+$, the marginal distribution of a step length s_k ($k = 1, .., n$) is calculated from that of the underlying Pearson–Dirichlet random walk to be:

$$p(s_k) = \frac{\Gamma(nq)}{\Gamma(q)\Gamma((n-1)q)} \int_{s_k}^{\infty} f(l) \left(\frac{s_k}{l} \right)^{q-1} \left(1 - \frac{s_k}{l} \right)^{(n-1)q-1} \frac{dl}{l} \tag{41}$$

The endpoint of a n-step walk is simply given by $\boldsymbol{r} = \boldsymbol{r}_n^{(d)} = \sum_{k=1}^{n} s_k \boldsymbol{u}_k^{(d)}$, where $\boldsymbol{u}_k^{(d)}$ is a unit vector of $\mathbb{R}^d$. When the parent Pearson–Dirichlet walk $PD(d,n,q)$ is HU, the pdf of the endpoint, $g_{d,n,q}(\boldsymbol{r})$, and that, $G_{d,n,q}(r)$, of the distance from the origin to the endpoint read:

$$\begin{cases} g_{d,n,q}(\boldsymbol{r}) = \dfrac{\Gamma(k/2)}{\pi^{d/2}\Gamma(\delta+1)} \times \displaystyle\int_r^\infty \frac{f(l)}{l^{d+2\delta}}(l^2-r^2)^\delta dl \\ G_{d,n,q}(r) = \dfrac{2r^{d-1}}{B(\delta+1,d/2)} \times \displaystyle\int_r^\infty \frac{f(l)}{l^{d+2\delta}}(l^2-r^2)^\delta dl \end{cases} \tag{42}$$

as deduced from (16) where k and δ are given in Table 1. In (42), $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function. The Pearson–Liouville walk inherits a simple geometrical representation from its HU parent. Its endpoint $\boldsymbol{r}_n^{(d)}$ is indeed the projection in $\mathbb{R}^d$ of a vector $\boldsymbol{r}_n^{(k)}$ of $\mathbb{R}^k$ whose stochastic representation is $\boldsymbol{r}_n^{(k)} \triangleq l\boldsymbol{u}^{(k)}$ where l is independent of the uniform unit vector $\boldsymbol{u}^{(k)}$. Therefore, we get from (37) that:

$$\boldsymbol{r}_n^{(d)} \triangleq \frac{l\boldsymbol{G}^{(d)}}{\sqrt{\|\boldsymbol{G}^{(d)}\|^2 + \chi_{k-d}^2}} \tag{43}$$

where l is independent of the random vector $\boldsymbol{G}^{(d)}$ and of the chi-square χ_{k-d}^2. A stochastic equation for the distance from the origin to the endpoint might similarly be written from (38). The previous representations are efficient for Monte-Carlo simulations of the endpoints of these walks. If we define now a vector $\boldsymbol{t}^{(k)}$ of $\mathbb{R}^k$, whose distribution is spherical with a density $p_k(t = \|\boldsymbol{t}^{(k)}\|)$ equal to the generating density $f(t)$ normalized by the area of the hypersphere of radius t in $\mathbb{R}^k$:

$$p_k(t) = \frac{\Gamma(k/2)f(t)}{2\pi^{k/2}t^{k-1}} \tag{44}$$

Then the d-dimensional projection $\boldsymbol{r} = \boldsymbol{r}_n^{(d)}$ of $\boldsymbol{t}^{(k)}$ has a density $g_{d,n,q}(\boldsymbol{r})$ given by (42). The latter result is consistently found by a direct application of relation (28) of [32]. The characteristic function of the distribution of $\boldsymbol{r}$, $\Phi_{d,n,q}(\rho) = \langle e^{i\boldsymbol{\rho}\cdot\boldsymbol{r}_n^{(d)}}\rangle$, depends only on the modulus $\rho = \|\boldsymbol{\rho}\|$ as $g_{d,n,q}(\boldsymbol{r})$ is spherically symmetric. It reads:

$$\Phi_{d,n,q}(\rho) = \langle e^{i\boldsymbol{\rho}\cdot\boldsymbol{r}}\rangle = \frac{2^{(k-2)/2}\Gamma(k/2)}{\rho^{(k-2)/2}} \int_0^\infty \frac{f(l)J_{(k-2)/2}(\rho l)}{l^{(k-2)/2}} dl \tag{45}$$

The pdf's $g_{d,n,q}(\boldsymbol{r})$ and $G_{d,n,q}(r)$ (42) can be derived alternatively from the characteristic function by the following inversion formula (equations (6) and (10) of [32]):

$$\begin{cases} g_{d,n,q}(\boldsymbol{r}) = \dfrac{1}{(2\pi)^{d/2}r^{(d-2)/2}} \times \displaystyle\int_0^\infty \rho^{d/2}J_{(d-2)/2}(r\rho)\Phi_{d,n,q}(\rho)d\rho \\ G_{d,n,q}(r) = \dfrac{r^{d/2}}{2^{(d-2)/2}\Gamma(d/2)} \times \displaystyle\int_0^\infty \rho^{d/2}J_{(d-2)/2}(r\rho)\Phi_{d,n,q}(\rho)d\rho \end{cases} \tag{46}$$

Equation (42) is consistently obtained when plugging (45) into (46), interchanging the order of integration and using integral 6.575.1 of [28].

8.2 An Example: Random Walks with i.i.d. Gamma Distributed Step Lengths

We consider now a Pearson-"gamma" random walk of n steps in $\mathbb{R}^d$ whose lengths are i.i.d. gamma distributed with a shape parameter q, denoted below as $PG(d,n,q)$. Then the characteristic function, $\Phi_{d,n,q}(\rho) = \langle e^{i\boldsymbol{\rho}\cdot\boldsymbol{r}_n^{(d)}}\rangle$, of the spherical distribution of the endpoint of this walk, $\boldsymbol{r}_n^{(d)} = \sum_{k=1}^n s_k \boldsymbol{u}_k^{(d)}$, is readily obtained to be:

$$\begin{cases} \Phi_{d,1,q}(\rho) \propto \dfrac{1}{\rho^q}\displaystyle\int_0^\infty x^{q-d/2} J_{(d-2)/2}(x)\exp(-x/\rho)dx \\ \Phi_{d,n,q}(\rho) = (\Phi_1^{(d)}(\rho))^n \end{cases} \tag{47}$$

and from integral 6.621.1 of [28]:

$$\Phi_{d,1,q}(\rho) = \frac{1}{(1+\rho^2)^{q/2}} F\left(\frac{q}{2}, \frac{d-q-1}{2}; \frac{d}{2}; \frac{\rho^2}{(1+\rho^2)}\right) \tag{48}$$

where $F(a,b;c;z)$ is a Gaussian hypergeometric function. When $q = d-1+\Delta$, with $\Delta = 0, 1$, the c.f. $\Phi_{d,n,d-1+\Delta}(\rho)$ simplifies to:

$$\Phi_{d,n,d-1+\Delta}(\rho) = \frac{1}{(1+\rho^2)^{n(d-1+2\Delta)/2}} \tag{49}$$

while it is:

$$\Phi_{d,n,(d-2)/2}(\rho) = \left[2\left(\sqrt{1+\rho^2}-1\right)/\rho^2\right]^{n(d-2)/2} \tag{50}$$

for $q = (d-2)/2$. The Fourier inversion of the c.f. given by (49) yields the pdf of the endpoint $g_{d,n,d-1+\Delta}(\boldsymbol{r})$ and thus the pdf of the distance $G_{d,n,d-1+\Delta}(r)$ of a "gamma" walk $PG(d,n,d-1+\Delta)$ in $\mathbb{R}^d$ $(d+\Delta \geq 2)$. These densities are expressed in terms of $K_v(x)$, a modified Bessel function of the second kind:

$$\begin{cases} v = (n(d-1+2\Delta)-d)/2 \\ g_{d,n,d-1+\Delta}(\boldsymbol{r}) = \dfrac{r^v K_v(r)}{2^{v+d-1}\pi^{d/2}\Gamma(n(d-1+2\Delta)/2)} \\ G_{d,n,d-1+\Delta}(r) = \dfrac{r^{v+d-1}K_v(r)}{2^{v+d-2}\Gamma(d/2)\Gamma(n(d-1+2\Delta)/2)} \end{cases} \tag{51}$$

The density $g_{d,n,d-1+\Delta}(r)$ and $\Phi_{d,n,d-1+\Delta}(\rho)$ (49), once properly normalized, are dual spherical densities [33, 34] related through Hankel transforms [32]. The latter density is that of a d-dimensional spherical Student distribution with v degrees of freedom [19]. As required, $g_{2,n,1}(\boldsymbol{r})$ (51) coincides with the density calculated by Stadje for a walk in 2D with exponentially distributed step lengths (equation (1.5) of [9]). The densities $g_{d,n,d-1}(r)$ and $G_{d,n,d-1}(r)$ given by (51), for $q = d-1$ $(\Delta = 0)$, were equally obtained from the densities of the parent HU walk of the first family with the method described in the previous subsection. It suffices to apply (42) with $\delta = (n(d-1)-(d+1))/2$ and to use integral 3.387.6 of [28].

The method of the previous subsection can be applied to the second HU family, with $q = (d-2)/2$ $(d \geq 3)$, $n \geq 2$ and $\delta = (n(d-2)-d)/2$ (Table 1). The endpoint and the distance pdf's read:

$$\begin{cases} v = (n(d-2)-d)/2 \\ g_{d,n,(d-2)/2}(\boldsymbol{r}) = \dfrac{n(d-2)}{2\pi^{d/2}\Gamma(v+1)} \times \displaystyle\int_r^\infty \frac{\exp(-l)}{l^{v+d/2+1}}(l^2-r^2)^v dl \\ G_{d,n,(d-2)/2}(r) = \dfrac{n(d-2)r^{d-1}}{\Gamma(d/2)\Gamma(v+1)} \times \displaystyle\int_r^\infty \frac{\exp(-l)}{l^{v+d/2+1}}(l^2-r^2)^v dl \end{cases} \tag{52}$$

Explicit general solutions, like those given by (51), were not found in that case. Precise numerical calculations of $G_{d,n,(d-2)/2}(r)$ can however be performed from the characteristic function ((46) and (50)), where $n \geq 3$ for $d = 2$. The latter distribution can be expressed explicitly in some specific cases, for instance for $d = 3$ and $n = 3, 5$:

$$\begin{cases} G_{3,3,1/2}(r) = 8r^2 erfc(\sqrt{r}) + \dfrac{2}{\sqrt{\pi}}\sqrt{r}(2-4r)\exp(-r) \\ G_{3,5,1/2}(r) = \dfrac{4}{3}r^2(4r^2-15)erfc(\sqrt{r}) + \dfrac{8}{3\sqrt{\pi}}r^{3/2}(6+r-2r^2)\exp(-r) \end{cases} \tag{53}$$

where $erfc(x)$ is the complementary error function.

For the previous "gamma" walk, $PG(d,n,q)$, the total travelled distance l after n steps is also gamma distributed, with a pdf given by $f(l) = l^{nq-1}\exp(-l)/\Gamma(nq)$. We consider now a Pearson–Dirichlet random walk $PD(d,n,q)$, not necessarily restricted to be hyperspherical uniform, whose endpoint has a pdf $p_{d,n,q}(r)$ with $l = 1$ and $p_{d,n,q}(r) = 0$ for $r > 1$. From an equation analogous to (42), the distance from the endpoint of the "gamma" walk $PG(d,n,q)$ to the origin is found to have a pdf, $G_{d,n,q}(r)$, which may be deduced by a Laplace transform from the pdf, $p_{d,n,q}(r)$, of the parent Pearson–Dirichlet walk, namely:

$$G_{d,n,q}(r) = \frac{2\pi^{d/2}r^{nq-1}}{\Gamma(d/2)\Gamma(nq)}\int_0^\infty \exp(-rt)t^{nq-d-1}p_{d,n,q}(1/t)dt \tag{54}$$

Conversely, an inverse Laplace transform might in principle yield $p_{d,n,q}(r)$ from $G_{d,n,q}(r)$ but such calculations are not necessarily straightforward.

Beghin and Orsingher [24] studied a planar random motion at finite constant velocity in which a particle changes direction at even-valued Poisson events ($q = 2$). They derived, among others, the densities of the particle position at time t given the number of reorientations between 0 and t. Interestingly, such densities are obtained as mixtures of pdf's of the motion of a particle changing direction at all Poisson events [24]. Their results solve thus the 2D Pearson–Dirichlet walk $PD(2,n,2)$, with a step length distribution deduced from $p(s_k) = s_k\exp(-s_k)$ $(k = 1,..,n)$. When the total number of Poisson events is odd, the pdf $p_{2,n,2}(r)$ $(n \geq 2)$ is indeed expressed as:

$$p_{2,n,2}(\boldsymbol{r}) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} r_j(h)\,p_{2,n+2h,1}(\boldsymbol{r}) \tag{55}$$

In (55), j is equal to 1 when the number of steps is odd, $n = 2p+1$ $(p \geq 1)$, to 3 when n is even, $n = 2p+2$ $(p \geq 0)$, and $\lfloor x \rfloor = floor(x)$. The densities $p_{2,n+2h,1}(r)$ are given by (1).

The weights $r_j(h)$ were calculated by Beghin and Orsingher (equation (3.3) of [24]):

$$\begin{cases} r_1(h) = \left(\dfrac{4p}{4p+3}\right)\dbinom{p}{h}\dfrac{B(2p, p+\frac{3}{2})}{B(p+h+1, p-h+\frac{3}{2})} \\ r_3(h) = \dfrac{3(2h+1)}{4(2p+3)^{w(h)}}\dbinom{p+1}{h}\dfrac{B(2p+\frac{5}{2}, p+\frac{3}{2})}{B(p+h+\frac{3}{2}, p-h+\frac{5}{2})} \qquad \left(h=0,..,\left\lfloor\dfrac{n}{2}\right\rfloor\right) \\ w(0)=0, \quad w(h) = 1+\left\lfloor\dfrac{h}{p+1}\right\rfloor (1 \le h \le p+1) \end{cases} \tag{56}$$

where $B(x, y)$ is the beta function. The pdf of the distance from the endpoint of the "gamma" walk $PG(2, n, 2)$ to the origin, $G_{2,n,2}(r)$, is given by (51) with $d=2$, $\Delta=1$, $q=2$. Inserting (55) into (54), it is possible to relate linearly the moments of the latter distance, $\langle r^{2x}\rangle = \int_0^\infty r^{2x} G_{2,n,2}(r)dr$, $x \ge 0$, to the weights $r_j(h)$, $j = 1$ or 3, by:

$$\sum_{h=0}^{\lfloor\frac{n}{2}\rfloor} r_j(h) \times \frac{\Gamma(\frac{n+1}{2}+h)}{\Gamma(\frac{n+1}{2}+h+x)} = 4^x \times \frac{\Gamma(\frac{3n}{2}+x)}{\Gamma(\frac{3n}{2})} \times \frac{\Gamma(2n)}{\Gamma(2n+2x)} \tag{57}$$

The weights calculated from the latter linear equations for small values of n, which agree with those given by (56), confirm the usefulness of (54).

The hyperspherical uniform property provides a convenient way of performing Monte-Carlo simulations of endpoints of "gamma" walks $PG(d, n, q)$ whose step lengths have gamma distributions with shape parameters respectively equal to $\frac{d}{i}-1$ $(d \ge i+1)$ for $i = 1, 2$. The stochastic representation, $\boldsymbol{r}_n^{(d)} \triangleq l\boldsymbol{G}^{(d)}/\sqrt{\|\boldsymbol{G}^{(d)}\|^2 + \chi^2_{k-d}}$, where l is then gamma distributed with a shape parameter $n(\frac{d}{i}-1)$, shows that it suffices to generate only $d+2$ independent random variables for any n. Many of the Pearson–Dirichlet HU walks, and in particular all uniform walks, and the "gamma" walks defined in Sect. 8.2 were further investigated by Monte-Carlo simulations to obtain "experimental" distributions of the distance of the endpoint to the origin. All results were found to be in excellent agreement with the corresponding closed-form distributions derived in the present work (Fig. 6).

9 Conclusions

We introduced a variant of the Pearson–Rayleigh random walk of $n \ge 2$ steps which are random vectors whose orientations are independent and uniform in $\mathbb{R}^d$ and whose lengths have a Dirichlet distribution whose parameters are all equal to a given positive number q. Two families of walks, named "hyperspherical uniform", are obtained for values of the parameter q equal to $\frac{d}{i}-1$, with $d \ge i+1$, for $i = 1, 2$ respectively. For any number of steps, the endpoint distributions of the latter walks are identical to the distributions of the projection in the walk space $\mathbb{R}^d$ of a point randomly chosen on the surface of the unit hypersphere of a hyperspace $\mathbb{R}^k$. The hyperspace dimension associated with each family is equal to $n(d-i)+i$. The associated probability density function of the endpoint position $\boldsymbol{r}$ is consequently given by $p_{d,n,q}(\boldsymbol{r}) \propto (1-r^2)^\delta$ where $2\delta = n(d-i)-(d+2-i)$. Four walks, two of two steps and two of three steps, whose endpoints are uniformly distributed in the inside of the unit hypersphere in $\mathbb{R}^d$ are then found for $d = 2, 3, 4$ from $\delta = 0$. The single uniform walks in 2D and in 4D were previously known [10, 12, 13], two new uniform walks in 3D are evidenced here. Three additional uniform walks are obtained from finite integrals of products of powers and Bessel functions of the first kind for Dirichlet distributions of step lengths whose parameters are not all the same. Once constructed, the HU walks may be used to derive the endpoint distributions of random walks whose total step length is distributed according to a given law.

Important aspects of the studied walks are connected with the general problem of the random fragmentation of the unit interval (see for instance [35] and references therein). Finally, there is a deep connection between the HU walks and finite integrals of products of Bessel functions and powers.

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Appendix: Uniform Distribution of a Vector over the Surface of the Unit Hypersphere in $\mathbb{R}^k$

Consider first a unit vector $\boldsymbol{u}^{(k)}$ whose tip spans uniformly the surface of the hypersphere in $\mathbb{R}^k$. Using hyperspherical coordinates, the characteristic function (c.f.) $\Omega_k(\boldsymbol{\rho}) = \langle e^{i\boldsymbol{\rho}.\boldsymbol{u}^{(k)}}\rangle$ of the distribution of $\boldsymbol{u}^{(k)}$, which depends only on the modulus $\rho = \|\boldsymbol{\rho}\|$, is found to be:

$$\Omega_k(\rho) = \frac{\Gamma(k/2)}{\sqrt{\pi}\Gamma((k-1)/2)} \int_0^{\pi} e^{i\rho\cos(\theta)} \sin^{k-2}(\theta) d\theta = \frac{2^{(k-2)/2}\Gamma(k/2)}{\rho^{(k-2)/2}} J_{(k-2)/2}(\rho) \quad (58)$$

where the c.f. has been expressed in term of a Bessel function of the first kind, $J_u(\rho)$. More generally, the c.f. $\Phi_k(\boldsymbol{\rho}) = \langle e^{i\boldsymbol{\rho}\cdot\boldsymbol{r}^{(k)}}\rangle$ of any spherically symmetric vector $\boldsymbol{r}^{(k)}$ of $\mathbb{R}^k$, whose distribution is invariant by any orthogonal transformation, is similarly a function of the sole modulus of $\boldsymbol{\rho}$ [19]. Thus, the c.f. of the marginal distribution of any number j $(j = 1, .., k)$ of components of a spherical vector $\boldsymbol{r}^{(k)}$ is still $\Phi_k(\rho)$ but ρ is now the modulus of a vector $\boldsymbol{\rho}$ in which $k - j$ components are made equal to zero. To prove that a spherically symmetric vector $\boldsymbol{r}_n^{(d)}$ is the projection of a unit vector $\boldsymbol{u}^{(k)}$, it would suffice to show that the c.f. of the first component r_1 of $\boldsymbol{r}_n^{(d)}$ is $\Phi_1(\rho) = \langle e^{i\rho\cdot r_1}\rangle = \Omega_k(\rho)$.

Consider now a vector $\boldsymbol{G}^{(k)} = (G_1, .., G_k)$ of $\mathbb{R}^k$, whose components are i.i.d. standard Gaussian variables with a zero mean and a variance of 1 and whose modulus is $G = \|\boldsymbol{G}^{(k)}\|$. Then the tip of the unit vector $\boldsymbol{u}^{(k)} = (u_1 = G_1/G, .., u_k = G_k/G)$ spans uniformly the surface of the hypersphere in $\mathbb{R}^k$ (see for instance [19] p. 20). Every G_i^2 $(i = 1, .., k)$ is gamma distributed with a shape parameter of $1/2$ and a scale parameter of $1/2$: $S_i = G_i^2$, $p(s_i) = s_i^{-1/2}\exp(-s_i/2)/\sqrt{2\pi}$ (it is a chi distribution with one degree of freedom [20]). The distribution of $(l_1 = u_1^2 = G_1^2/G^2, .., l_k = u_k^2 = G_k^2/G^2)$ is consequently a Dirichlet distribution whose parameters are all equal to $1/2$. Then, the joint distribution of any number j of components of $\boldsymbol{u}^{(k)}$ can be obtained by using the amalgamation property (Sect. 2.2). The resulting pdf is given by [19, 26]:

$$p_j(u_1, u_2, .., u_j) = \frac{\Gamma(k/2)}{\Gamma((k-j)/2)\pi^{j/2}} \left(1 - \sum_{i=1}^{j} u_i^2\right)^{(k-j-2)/2} \quad \left(\sum_{i=1}^{j} u_i^2 < 1\right) \quad (59)$$

Thus, a uniform distribution in the inside of a $j = k - 2$-dimensional hypersphere of radius R can be derived from the projection of a uniform distribution over the surface of a k-dimensional hypersphere of radius R. The latter property was used by Lord to obtain the pdf of the distance between two points uniformly distributed in a hypersphere of $\mathbb{R}^{k-2}$ from a Pearson random walk of two steps of length R in $\mathbb{R}^k$ [27]. The Dirichlet distribution of $(l_1 = u_1^2, .., l_k = u_k^2)$, whose parameters are all equal to $1/2$ and (7) with $\boldsymbol{\alpha}^{(k)} = (1/2, .., 1/2)$ and $\boldsymbol{\beta}^{(k)} = (p, 0, \ldots, 0)$ yield the even moments $\langle u_i^{2p} \rangle$ of a single component of $\boldsymbol{u}^{(k)}$:

$$\langle u_i^{2p} \rangle = \frac{(1/2)_p}{(k/2)_p} = \frac{(2p-1)!!}{\prod_{j=1}^{p}(k+2j-2)} \tag{60}$$

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